The Poisson Distribution.

Recall the binomial distr-n: we have the random variable X which takes values $\{0, 1, \dots, n\}$ with the pdf

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

If n and k are large (1000 and 500 for instance) the binomial coeff. $\binom{n}{k}$ is hard to evaluate. Thus, we need some easy-to-use approximation. One of the first such approx-ns was given by Poisson.

Thm1 (Poisson) Suppose X is a binomial random variable. If $n \to \infty$ and $p \to 0$ in such a way that $\lambda = np$ remains constant, then $\lim\limits_{\substack{n\to\infty \\ p\to 0 \\ \lambda = np}} P(X=k) = \lim\limits_{\substack{n\to\infty \\ p\to 0 \\ \lambda = np}} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}.$

Proof: $$\lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} =$$

$$= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \lambda^k \cdot \frac{1}{n^k} \left(1-\frac{\lambda}{n}\right)^{-k} \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!} \cdot \frac{1}{(n-\lambda)^k} \left(1-\frac{\lambda}{n}\right)^n$$

Recall that $\lim\limits_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$, so it remains to show that $\lim\limits_{n\to\infty} \frac{n!}{(n-k)!} \cdot \frac{1}{(n-\lambda)^k} = 1.$

$$\lim_{n\to\infty} \frac{n!}{(n-k)!} \cdot \frac{1}{(n-\lambda)^k} = \lim_{n\to\infty} \frac{n \cdot \ldots \cdot (n-k+1)}{(n-\lambda) \ldots (n-\lambda)} = 1$$ (the limit is equal to the ratio of leading coeffs).

Def-n. A random variable X is said to have a Poisson distr-n if $p_X(k) = P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$ with $k \in \mathbb{Z}_{\geq 0} = \{0,1,2,\dots\}$ and λ is a positive constant.

Thm2. (0) $P_X(k)$ defines a pdf.

(1) $\mathbb{E}(X) = \lambda$

(2) $Var(X) = \lambda$.

Proof: (0) clearly $p_X(k) \geq 0$ for all $k \in \mathbb{Z}_{\geq 0}$, also

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

(1)* Let $\frac{\partial}{\partial_\lambda}$ be the differential operator, which acts on f-ns by taking the derivative with respect to λ, i.e. $\frac{\partial f(\lambda)}{\partial_\lambda} = f'(\lambda)$.

By def-n $\mathbb{E}(X) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \cdot k$.

On the other hand, we claim that $\mathbb{E}(X) =$

$$= \left(\lambda \frac{\partial}{\partial_\lambda} + \lambda\right) \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda}\right) \quad (*)$$

Let us check that $(*)$ holds:

$$\left(\lambda \frac{\partial}{\partial_\lambda} + \lambda\right)\left(\frac{\lambda^k}{k!} e^{-\lambda}\right) = \lambda \frac{\partial}{\partial_\lambda}\left(\frac{\lambda^k}{k!} e^{-\lambda}\right) + \frac{\lambda^{k+1}}{k!} e^{-\lambda} =$$

$\lambda\left(k\cdot\frac{\lambda^k}{k!}e^{-\lambda}-\frac{\lambda^k}{k!}e^{-\lambda}\right)+\frac{\lambda^{k+1}}{k!}e^{-\lambda}=\frac{\lambda^k}{k!}e^{-\lambda}\cdot k-\frac{\lambda^{k+1}}{k!}e^{-\lambda}+\frac{\lambda^{k+1}}{k!}e^{-\lambda}$

$=\frac{\lambda^k}{k!}e^{-\lambda}\cdot k$. As the computation above is valid ~~holds~~ for any $k\geq 0$, $(*)$ holds.

Now, since $\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}=1$ (as checked in (0)), $(*)$ gives $\mathbb{E}(X)=\left(\lambda\frac{\partial}{\partial_\lambda}+\lambda\right)\cdot 1=0+\lambda=\lambda$.

(2)* It is analogous to check

$$\mathbb{E}(X^2)=\left(\lambda\frac{\partial}{\partial_\lambda}+\lambda\right)\left(\lambda\frac{\partial}{\partial_\lambda}+\lambda\right)\left(\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}\right)\quad (**)$$

$$\mathbb{E}(X^2)=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}\cdot k^2$$

$(**)$ gives $\mathbb{E}(X^2)=\left(\lambda\frac{\partial}{\partial_\lambda}+\lambda\right)\left(\lambda\frac{\partial}{\partial_\lambda}+\lambda\right)\cdot 1=\left(\lambda\frac{\partial}{\partial_\lambda}+\lambda\right)\cdot\lambda=$
$=\lambda+\lambda^2$.

Finally, $Var(X)=\mathbb{E}(X^2)-\mathbb{E}^2(X)=\lambda+\lambda^2-\lambda^2=\lambda$.

<u>Exercise</u>. Give a proof of (1) & (2) in Thm 2 using the mgf of Poisson distr-n.

Suppose a series of events is occuring during a time interval T (length). Divide $[0;T)$ into n subintervals of length $\frac{T}{n}$ (assume n is large). Furthemore, suppose that

1. The probability of two or more events occurring in the same subinterval is 0 (i.e. we have a Bernoulli r.v. on each subinterval)
2. The probability of occurrence in one subinterval is independent of one in another.
3. The probability that an event occurs during a given (length) time subinterval is constant over $[0;T)$.

(1)-(3) say that we have the Binomial (as sum of Bernoulli) distr-n over $[0;T)$ and as $n\to\infty$ this tends to Poisson distr-n (Thm 1) with mean np

Examples (HW, page 22)

(2) Suppose a bus arrives every 10 mins (on average).
If this is a Poisson process:
(a) find the probability that there will be 3 buses in the next 15 mins.

Answer: $\lambda = 1.5$ buses / 15 mins

$$P(X=3) = \frac{(1.5)^3}{3!} e^{-1.5}$$

(b) Find the probability that you have to wait at least 10 mins.

$\lambda = 1 \text{ bus} / 10 \text{ mins}$

Answer: $P(X=0) = \frac{1^0}{0!} e^{-1} = \frac{1}{e}$.

(c) What is the expected time until 5 buses go by?

Since $\mathbb{E}(X) = \lambda = 1$ bus in 10 mins, for $\tilde{\lambda} = 5 \text{ buses} / 50 \text{mins}$

$\mathbb{E}(\tilde{X}) = 5 \Rightarrow 50$ mins.

#5. Naturalists are tagging sharks. From past experience they have found that they find (and tag) an average of 1 shark per 2 hours. Assume this is a Poisson process.

(a) Find the probability that at least 2 sharks arrive in the next three hours.

Sol-n: $\lambda = 1.5 / 3$ hours.

$$P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - e^{-1.5}\left(1 + \frac{1.5^1}{1!}\right)$$

$$= 1 - 2.5e^{-1.5} \approx 0.44.$$

(b) The naturalists would like to tag two sharks but only have three hours. They will work until either they tag two sharks or the three hours are over. Find the expected number of sharks they tag.

Sol-n: notice that they can tag at most 2 sharks, hence,

$\mathbb{E}(\text{sharks tagged}) = P(X=0) \cdot 0 + P(X=1) \cdot 1 + P(X \geq 2) \cdot 2 \approx 1.21.$

(if $X > 2$, they stop earlier)